

A Dichotomy Theorem for Circular Colouring Reconfiguration

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Abstract

Let p and q be positive integers with $p/q \geq 2$. The “reconfiguration problem” for circular colourings asks, given two (p, q) -colourings f and g of a graph G , is it possible to transform f into g by changing the colour of one vertex at a time such that every intermediate mapping is a (p, q) -colouring? We show that this problem can be solved in polynomial time for $2 \leq p/q < 4$ and that it is PSPACE-complete for $p/q \geq 4$. This generalizes a known dichotomy theorem for reconfiguring classical graph colourings. As an application of the reconfiguration algorithm, we show that graphs with fewer than $(k-1)!/2$ cycles of length divisible by k are k -colourable.

1 Introduction

In recent years, a large body of research has emerged concerning so called “reconfiguration” variants of combinatorial problems (see, e.g., the survey of van den Heuvel [14] and the references therein, and well as [15, 16]). These problems are typically of the following form: given two solutions to a fixed combinatorial problem (e.g. two cliques of order k in a graph or two satisfying assignments of a specific 3-SAT instance) is it possible to transform one of these solutions into the other by applying a sequence of allowed modifications such that every intermediate object is also a solution to the problem?

As a specific example, for a fixed integer k and a graph G , one may ask the following: given two (proper) k -colourings f and g of G , is it possible to transform f into g by changing

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the colour of one vertex at a time such that every intermediate mapping is a k -colouring?¹ In the affirmative we say that f *reconfigures* to g . This problem is clearly solvable in polynomial time for $k \leq 2$. Rather surprisingly, Cereceda, van den Heuvel and Johnson [7] proved that it is also solvable in polynomial time for $k = 3$ despite the fact that determining if a graph admits a 3-colouring is NP-complete. On the other hand, Bonsma and Cereceda [3] proved that, when $k \geq 4$, the problem is PSPACE-complete. (As pointed out in [6], a similar result was proved by Jakob [17], but in his result the integer k is part of the input.) Combining these two results yields the following dichotomy theorem:

Theorem 1 (Cereceda, van den Heuvel and Johnson [7]; Bonsma and Cereceda [3]). *The reconfiguration problem for k -colourings is solvable in polynomial time for $k \leq 3$ and is PSPACE-complete for $k \geq 4$.*

In this paper, we study the complexity of the reconfiguration problem for circular colourings. Given a graph G and positive integers p and q with $p/q \geq 2$, a *(circular) (p, q) -colouring* of G is a mapping $f : V(G) \rightarrow \{0, \dots, p-1\}$ such that

$$\text{if } uv \in E(G), \text{ then } q \leq |f(u) - f(v)| \leq p - q. \quad (1)$$

Clearly, a $(p, 1)$ -colouring is nothing more than a p -colouring and so (p, q) -colourings generalize classical graph colourings. Circular colourings were introduced by Vince [22], and have been studied extensively; see the survey of Zhu [25]. Analogous to that of classical graph colourings, the reconfiguration problem for circular colourings asks, given (p, q) -colourings f and g of G , whether it is possible to reconfigure f into g by recolouring one vertex at a time while maintaining (1) throughout.

Classical graph colourings and circular colourings are both special cases of graph homomorphisms. Recall, a *homomorphism* from a graph G to a graph H (also called an *H -colouring* of G) is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. The notation $f : G \rightarrow H$ indicates that f is a homomorphism from G to H . In this language, a k -colouring is simply a homomorphism to a complete graph on k vertices. A (p, q) -colouring of G is equivalent to a homomorphism from G to the graph $G_{p,q}$ which has vertex set $\{0, \dots, p-1\}$ and edge set $\{ij : q \leq |i - j| \leq p - q\}$. The graph $G_{p,q}$ is called a *circular clique*.

Remark 2. It is well known that $G_{p,q}$ admits a homomorphism to $G_{p',q'}$ if and only if $p/q \leq p'/q'$ [22]. Therefore, since the composition of two homomorphisms is a homomorphism, a graph G admits a (p, q) -colouring if and only if it admits a (p', q') -colouring for all $p'/q' \geq p/q$.

Given two homomorphisms $f, g : G \rightarrow H$, we say f *reconfigures* to g if there a sequence $(f = f_0, f_1, f_2, \dots, (f_n = g))$ of homomorphisms from G to H such that f_i and f_{i+1} differ on only one vertex. The sequence is referred to as a *reconfiguration sequence*. Clearly the existence of a reconfiguration sequence from f to g can be determined independently for each component of G , so we may assume that G is connected. We define the general homomorphism reconfiguration problem as follows. Let H be a fixed graph.

¹Throughout the paper, the term *k -colouring* will refer to a proper k -colouring.

H -Recolouring

Instance: A connected graph G , and two homomorphisms $f, g : G \rightarrow H$.

Question: Does f reconfigure to g ?

When $H = K_k$ or $H = G_{p,q}$ we will call the problem k -RECOLOURING and (p, q) -RECOLOURING respectively. Thus, Theorem 1 is a dichotomy theorem for k -RECOLOURING. Our main result is a dichotomy theorem for (p, q) -RECOLOURING:

Theorem 3. *Let p, q be fixed positive integers with $p/q \geq 2$. Then the (p, q) -RECOLOURING problem is solvable in polynomial time for $2 \leq p/q < 4$ and is PSPACE-complete for $p/q \geq 4$.*

The complexity of H -RECOLOURING is only known for a handful of families of targets. Theorem 1 is a dichotomy theorem for the family of complete graphs and Theorem 3 is a dichotomy theorem for the family of circular cliques. Recently, Wrochna [24] (see also [23]) proved that H -RECOLOURING is polynomial whenever H does not contain a 4-cycle. In contrast, one can observe that $G_{p,q}$ contains 4-cycles whenever $p > 2q + 1$ and so the polynomial side of Theorem 3 does not follow directly from the result of Wrochna. In a follow-up paper [5], we determine the complexity of H -RECOLOURING for several additional classes of graphs including, for example, odd wheels.

The rest of the paper is outlined as follows. In Section 2, we provide an explicit polynomial-time algorithm for deciding the (p, q) -RECOLOURING problem when $2 \leq p/q < 4$. In Section 3, we show that, when $p/q \geq 4$, the reconfiguration problem for $\lfloor p/q \rfloor$ -colourings can be reduced to the reconfiguration problem for (p, q) -colourings, thereby completing the proof of Theorem 3 (via Theorem 1). We close the paper by presenting an unpublished argument of Wrochna which uses a result of [7] (on which our algorithm is based) to show that graphs with no cycle of length $0 \bmod 3$ are 3-colourable. This result was originally proved by Chen and Saito [8]. We then generalize Wrochna’s argument to show that graphs with chromatic number greater than k must contain at least $\frac{(k-1)!}{2}$ distinct cycles of length $0 \bmod k$ and conjecture a stronger bound.

2 The Polynomial Cases: $2 \leq p/q < 4$

We now extend the ideas of [7] to study the complexity of (p, q) -RECOLOURING for $2 \leq p/q < 4$. Given two 3-colourings f and g of a graph G , the algorithm in [7] consists of two phases. The first phase tests whether the so-called “fixed vertices” (vertices that cannot be recoloured under *any* sequence of recolourings) are assigned the same colours under f and g . If not, then we know that f does not reconfigure to g and the algorithm terminates.

If the algorithm does not terminate at this point, then it enters the second phase. In this phase, the algorithm obtains two edge labellings of G from the vertex colourings f and g . The key property of these labellings is that, given that the algorithm did not terminate after the first phase, one can show that f reconfigures to g if and only if every cycle of G

has the same “weight” under both edge labellings. After a polynomial number of steps, the algorithm either produces a reconfiguration sequence or discovers a cycle with different weights under f and g . We follow a similar approach, but we begin by examining the edge relabelling problem which turns out to depend only on cycle weights.

Remark 4. The work in [7] has been extended in [18] where the authors find shortest paths between 3-colourings. In our work, we do not consider shortest paths, but rather are simply concerned with testing the existence of paths in polynomial time.

2.1 Edge Relabellings

Let f be a (p, q) -colouring of a graph G . Let \vec{G} be an oriented graph obtained by arbitrarily orienting each edge of G . (This orientation is required to define the weighting of paths, cycles, and cuts below. Thus throughout this section we will assume a graph G has an orientation \vec{G} .) The *edge-labelling induced by f* is defined as $\varphi_f : E(G) \rightarrow \{0, 1, \dots, p-1\}$ by $\varphi_f(e) = f(v) - f(u) \bmod p$ where $e = \vec{uv}$. Let C be a cycle of G and arbitrarily choose a direction of traversal. Let $E^+(C)$ be those edges of C whose orientation in \vec{G} agrees with the direction of traversal (forward arcs) and $E^-(C)$ be those edges of C whose orientation in \vec{G} is reversed to the direction of traversal (backward arcs). Define

$$\varphi_f(C) = \sum_{e \in E^+(C)} \varphi_f(e) + \sum_{e \in E^-(C)} (p - \varphi_f(e)). \quad (2)$$

Note that the summation above is in \mathbb{Z} , i.e. it is not reduced modulo p . The following properties of φ_f are immediate from the fact that f is a (p, q) -colouring and the sum $\varphi_f(C)$ is a telescoping series in the values of f on C .

P1 $q \leq \varphi_f(e) \leq p - q$ for all edges e .

P2 $\varphi_f(C) \equiv 0 \pmod{p}$ for all cycles C .

Given a graph G , an edge-labelling $\psi : E(G) \rightarrow \{0, 1, \dots, p-1\}$ is a (p, q) -labelling if it satisfies properties **P1** and **P2** above. Similar to the weighting function for cycles in (2), we define a weighting for paths: $\varphi(P)$ is the sum of $\varphi(e)$ and $p - \varphi(e)$ for forward and backward arcs e in P , respectively, according to some chose direction of traversal on P .

We now introduce the reconfiguration process for edge-labellings. Let G be a graph and \vec{G} an orientation of G . For $\emptyset \neq X \subsetneq V(G)$, the *edge cut* $\partial(X)$ is the set of edges with one end in X and the other in \bar{X} . Let $\partial^+(X)$ (resp. $\partial^-(X)$) be those edges oriented from X to \bar{X} (resp. \bar{X} to X) in \vec{G} . Given a (p, q) -labelling $\varphi : E(G) \rightarrow \{0, 1, \dots, p-1\}$, let α be an integer $1 \leq \alpha \leq p-1$, where $\varphi(e) \geq q + \alpha$ for $e \in \partial^+(X)$ and $\varphi(e) \leq p - q - \alpha$ for $e \in \partial^-(X)$. The labelling φ' obtained from φ by *relabelling on $\partial(X)$ by α* is defined by:

$$\varphi'(e) = \begin{cases} \varphi(e) - \alpha & \text{if } e \in \partial^+(X), \\ \varphi(e) + \alpha & \text{if } e \in \partial^-(X) \end{cases}$$

This process is called an *edge cut relabelling*. Clearly if φ is a (p, q) -labelling, then φ' is as well. It is easy to verify that for any cycle C , $\varphi(C) = \varphi'(C)$.

We remark an alternative relabelling definition is to simply require that if φ is a (p, q) -labelling, then φ' is as well (with no requirement that $\varphi(e) \geq q + \alpha$ for $e \in \partial^+(X)$ and $\varphi(e) \leq p - q - \alpha$ for $e \in \partial^-(X)$). In this case one would naturally reduce φ' modulo p after shifting by α . With this more general definition the weight of a cycle can change after relabelling. For example, consider a directed 4-cycle with $p = 4$ and $q = 1$. The constant labelling of 1 on each edge is a $(4, 1)$ -labelling. The weight of this cycle is 4 (when traversed in the direction of the edges). Using $\alpha = 2$, one can relabel to obtain the constant labelling $\varphi'(e) = 3$ for all edges: simply add 2 to the first and third edges, and -2 to the second and fourth edges. After reducing the weights modulo 4, we see $\varphi'(e) = 3$ for all edges. The weight of the cycle is now 12. Nonetheless, the two notions of relabelling are equivalent with the assumption $p/q < 4$ and the natural restriction $-p/2 \leq \alpha \leq p/2$. We will use the first definition as it eases the analysis below.

Theorem 5. *Let G be a connected graph and φ, ψ be two (p, q) -edge-labellings of G . Then φ reconfigures to ψ through a sequence of edge cut relabellings if and only if $\varphi(C) = \psi(C)$ for all cycles C in G .*

Proof. Relabelling on an edge cut does not change the weight of any cycles. Thus, if φ reconfigures to ψ , then $\varphi(C) = \psi(C)$ for all cycles C is a necessary condition. To prove sufficiency, assume $\varphi(C) = \psi(C)$ for each cycle C . We may assume $\varphi \neq \psi$. Let \overrightarrow{wu} be an arc such that $\varphi(wu) < \psi(wu)$. (The case where $\varphi(wu) > \psi(wu)$ is analogous.) Let T be a spanning tree of G rooted at u with all arcs directed away from u . For each $v \in V(G)$ define $\text{wt}(v, \varphi) = \varphi(P)$ where P is the (u, v) path in T , traversed from u to v . Similarly define $\text{wt}(v, \psi)$.

Let $X = \{v : \text{wt}(v, \varphi) \leq \text{wt}(v, \psi)\}$. Thus, $\overline{X} = \{v : \text{wt}(v, \varphi) > \text{wt}(v, \psi)\}$. Clearly $u \in X$ and it is easy to see $w \in \overline{X}$. Suppose to the contrary $w \in X$. Then uw is not an edge of T and thus the uw path in T plus the edge wu is a cycle C . Moreover this cycle must have $\varphi(C) < \psi(C)$ when traversed with the direction of the path P , contrary to our assumption. Hence $\emptyset \neq X \subsetneq V(G)$ and $\partial(X)$ is a well defined edge cut.

We claim that $\partial^+(X)$ must only contain arcs where $\varphi(e) > \psi(e)$ and arcs in $\partial^-(X)$ must have $\varphi(e) < \psi(e)$. Suppose to the contrary there is an edge $e = \overrightarrow{xy}$, $x \in X$, $y \in \overline{X}$ such that $\varphi(e) \leq \psi(e)$. (The proof for arcs in $\partial^-(X)$ analogous.) By construction there is a (u, x) -path P_1 such that $\varphi(P_1) \leq \psi(P_1)$. Thus $P_1 + y$ is an (u, y) -path such that $\varphi(P_1 + y) \leq \psi(P_1 + y)$. On the other hand, since $y \in \overline{X}$, the path (u, y) path in T , say P_2 , has the property that $\varphi(P_2) > \psi(P_2)$. Hence, reverse of the path is a (y, u) -path, P_2^R such that $\varphi(P_2^R) < \psi(P_2^R)$. The concatenation of $P_1 + y$ and P_2^R is a closed (u, u) -walk with less weight under φ than ψ . In particular, it must contain a cycle C such that $\varphi(C) \neq \psi(C)$, a contradiction. Let $\alpha = \min_{e \in \partial(X)} \{|\varphi(e) - \psi(e)|\}$. Relabel $\partial(X)$ by α to obtain φ' . Now φ' agrees with ψ on more edges than φ . The result follows. \square

Observe the second half of the proof of Theorem 5 shows that if there is a cycle C with different weights under φ and ψ , then C is the fundamental cycle in $T + e$ where $e \in \partial(X)$.

The discovery of such a cycle through examining fundamental cycles plays a key role in our polynomial time algorithm.

Corollary 6. *Let G be a graph together with two (p, q) -edge-labellings φ and ψ . Either φ reconfigures to ψ through a sequence of edge cut relabellings or there is a cycle C for which $\varphi(C) \neq \psi(C)$. Moreover, in the latter case C may be taken to be the fundamental cycle in $T + e$ where T is a fixed spanning tree of G , and $e \in \partial(X)$ as defined in the proof of Theorem 5. Determining which of the two cases (exclusively) holds can be determined in polynomial time.*

2.2 Vertex Recolouring

We now return to recolouring vertices. We have already observed that given a (p, q) -colouring f of a graph, we can construct a (p, q) -edge labelling φ_f . Conversely given φ , a (p, q) -edge labelling, using the ideas from the proof of Theorem 5, we can generate a weight, $\text{wt}(v, \varphi)$, for each vertex v . Reducing these weights modulo p yields a (p, q) -colouring f where $\varphi = \varphi_f$. The following proposition is immediate. Throughout this section addition (for shifting colours) is done modulo p .

Proposition 7. *Given a connected graph G , an edge-labelling $\psi : E(G) \rightarrow \{0, 1, \dots, p-1\}$ is a (p, q) -labelling if and only if $\psi = \varphi_f$ for some $f : G \rightarrow G_{p,q}$. Moreover, if $\varphi_f = \varphi_g$ for $f, g : G \rightarrow G_{p,q}$, then there exists $k \in \{0, 1, \dots, p-1\}$ such that $f(v) = g(v) + k$ for all $v \in V(G)$.*

To connect our work on edge cut relabellings back to vertex recolourings, we must realize edge cut relabelling through vertex recolourings where we recolour one vertex at a time. First, we observe that relabelling φ on $\partial(X)$ by α corresponds to the following vertex recolouring where we recolour an entire set of vertices (simultaneously). Let f' be the colouring obtained from f by:

$$f'(v) = \begin{cases} f(v) + \alpha & \text{if } v \in X, \\ f(v) & \text{otherwise.} \end{cases}$$

Then $\varphi_{f'}$ is the edge-labelling obtained from φ_f by relabelling on $\partial(X)$ by α . We call this process *recolouring the vertex set X by α* . Provided $\varphi(e) \geq q + \alpha$ for all $e \in \partial^+(X)$ and $\varphi(e) \leq p - q - \alpha$ for all $e \in \partial^-(X)$, the new colouring f' is a proper (p, q) -colouring. The following corollary is immediate from Proposition 7 and Theorem 5.

Corollary 8. *Let f and g be two (p, q) -colourings of a graph G . Then there is a sequence of colourings f, f_1, f_2, \dots, f_n , each obtained from its predecessor by a vertex set recolouring, where $f_n(v) = g(v) + k$ for all vertices v and some constant k if and only if $\varphi_f(C) = \varphi_g(C)$ for all cycles C in G .*

To complete our algorithm, we need to determine when recolouring a vertex set X by α can be realized by a sequence of single vertex recolourings. We begin by proving that for $p/q < 4$, recolouring by α can always be achieved through recolouring X by 1 (α times).

Note this not the case when $p/q \geq 4$. For example, consider a $(4, 1)$ -colouring of C_4 where the vertices are coloured $0, 1, 2, 3$ (in their cyclic order). The vertex with colour 0 can be recoloured to 2 . However, the recolouring cannot be done by increasing the colour by 1 (twice).

Given a pair $a, b \in \{0, \dots, p-1\}$, the *interval* $[a, b]$ is defined to be the set $\{a, a+1, \dots, b-1, b\}$ (where addition is modulo p). An important property of $G_{p,q}$ used in the proof below is that, when $2 \leq p/q < 4$, the common neighbours of any two vertices form an interval. This fact, and its proof, appear in [4]. (Roughly, if x and y are common neighbours in two non-contiguous intervals, then the non-neighbours of x and the non-neighbours of y are disjoint intervals. This requires $p \geq 2 + 2 \cdot (2q - 1)$ or $p/q \geq 4$.)

Proposition 9. *For $2 \leq p/q < 4$, let f, g be two (p, q) -colourings of a graph G where g is obtained from f by recolouring the vertex set X by α . Then there is a sequence of colourings $(f = f_0), f_1, f_2, \dots, (f_\alpha = g)$ where each colouring is obtained from its predecessor by recolouring X by 1 (for the entire sequence) or each is obtained from its predecessor by recolouring X by -1 (for the entire sequence).*

Proof. Let $v \in X$ and vu be an edge. If $u \in X$, then $(f(v) + 1)(f(u) + 1)$ and $(f(v) - 1)(f(u) - 1)$ are both edges of $G_{p,q}$. Consequently we can increment the colour by 1 of each vertex in X , or decrement the colour by 1 of each vertex in X , and maintain a proper (p, q) -colouring on X .

On the other hand, consider a neighbour u of v such that $u \notin X$. Note $f(u)f(v)$ and $f(u)(f(v) + \alpha)$ are both edges of $G_{p,q}$. If $\alpha = p/2$, then $f(u)$ is distance at least q from both $f(v)$ and $f(v) + p/2$ which implies $p/2 \geq 2q$, contrary to our assumption. If $\alpha < p/2 < 2q$, then $f(u)$ must belong to the interval $[f(v) + \alpha, f(v)]$. Thus $f(u)$ is adjacent in $G_{p,q}$ to every vertex of $[f(v), f(v) + \alpha]$ and in particular to $f(v) + 1$. Note this argument depends only on $\alpha < p/2$. Hence the colouring f_1 obtained from f by increasing the colour of each vertex in X by 1 is a (p, q) -colouring of G .

A similar argument shows for $\alpha > p/2$, the colouring f_1 obtained from f by decreasing the colour of each vertex in X by 1 is a (p, q) -colouring. The result follows. \square

Given G , an orientation \vec{G} , and an (p, q) -edge-labelling, say φ , let D_φ be the subdigraph of \vec{G} induced by edges with label q or $p - q$. Without loss of generality we may assumed D_φ is oriented so that all edges have label q . (To simplify notation we shall write D_f instead of D_{φ_f} when the edge-labelling φ_f comes from some vertex colouring f .) In the case $p/q = 2$, each edge will be oriented both ways, thus we have a directed 2-cycle on each edge.

Lemma 10. *Let f be a (p, q) -colouring of a graph G . Let f' be the (p, q) -colouring obtained from f by recolouring some vertex set X by 1 . Then f' can obtain from f by a sequence of recolourings of single vertices if and only if induced subdigraph $D_f[X]$ contains no directed cycles.*

Proof. Suppose $D_f[X]$ contains no directed cycles. We topologically sort the vertices of X to obtain an ordering x_1, x_2, \dots, x_t where $e = \overrightarrow{x_i x_j} \in E(D_f[X])$ implies $i < j$. For each $i = t, t-1, \dots, 1$ increase $f(x_i)$ by 1 .

On the other hand, suppose $D_f[X]$ contains a directed cycle. If one could sequentially increase the colour of each vertex in X by 1, then there would be a first vertex c_i in the cycle to have its colour changed. However, c_i has an out neighbour c_{i+1} (in $D_f[X]$) meaning $f(c_{i+1}) - f(c_i) = q$. The resulting colouring gives the edge $c_i c_{i+1}$ a weight of $q - 1$, i.e. the resulting colouring is not a proper (p, q) -colouring. \square

2.3 Polynomial Time Algorithm

We now describe the polynomial time algorithm for recolouring. We present a proof of correctness for each step below. The algorithm itself is in Figure 2.3.

Let f and g be two (p, q) -colourings of a graph G . As defined in [7], vertices whose colours can never change under any sequence of recolourings are called *fixed*. Following the ideas of [7] and using the results above, we identify (the only) three obstructions preventing the recolouring of f to g : fixed vertices, cycle weights, and weights of paths between fixed vertices.

The first step of the algorithm is to identify fixed vertices and verify $f(v) = g(v)$ for all fixed vertices.

Lemma 11. *Let f be a (p, q) -colouring of a graph G where $2 \leq p/q < 4$. Suppose v is a vertex in a strongly connected component of D_f . Then the colour of v cannot be changed under any sequence of recolourings.*

Proof. Since v belongs to a strongly connected component of D_f , v must belong to a directed cycle in D_f . By definition of D_f , the predecessor and successor of v on this cycle receive colours $f(v) - q$ and $f(v) + q$ respectively. As noted above, and proved in [4], the common neighbours of two vertices in $G_{p,q}$ form an interval. As the colour of v must be a common neighbour of $f(v) - q$ and $f(v) + q$, the only choice for v is $f(v)$. That is, the colour of v cannot change until the colour of one of its neighbours changes. However, this is true for every vertex on the directed cycle. All the vertices in the strongly connected component are fixed. \square

The strongly connected components of D_f can be found in linear time; hence, we can find the fixed vertices and verify equality of f and g on the fixed vertices in linear time. (See, e.g., [1].)

In Step 2 (a) of the algorithm we construct a spanning tree T rooted at a vertex u , and construct the cut $\partial(X)$ as in the proof of Theorem 5. Corollary 6 shows if some cycle has different weight under φ_f and φ_g , then we will discover a fundamental cycle in $T + e$ for some edge $e \in \partial(X)$.

In Step 2 (b) we calculate the shift α that will increase the number of edges upon which φ_f and φ_g agree. Proposition 9, Lemma 10, and Lemma 12 show that the vertices of X or \overline{X} can be recoloured 1 at a time, or there is a path between fixed vertices certifying that f does not recolour to g . Specifically, in the case we wish to increase the vertex set X by 1 and $D_f[X]$ contains a directed cycle, we can achieve the same change to the edge labelling on $\partial(X)$ by decreasing \overline{X} by 1 provided $D_f[\overline{X}]$ does not contain a directed cycle. In the event

both subgraphs contain a directed cycle, we have an obstruction to recolouring as described in the following lemma.

Lemma 12. *Let G be a graph with two (p, q) -colourings f and g . Suppose T is a spanning tree rooted at u . Let $X = \{v : \text{wt}(v, \varphi_f) \leq \text{wt}(v, \varphi_g)\}$ (as defined in Theorem 5). If $D_f[X]$ and $D_f[\overline{X}]$ both contain a directed cycle, then f does not recolour to g .*

Proof. Let x belong to a directed cycle in X and y belong to a directed cycle in \overline{X} . Then there are paths in T from u to x , say P_1 , and from u to y , say P_2 such that $\varphi_f(P_1) \leq \varphi_g(P_1)$ and $\varphi_f(P_2) > \varphi_g(P_2)$. Thus the reverse of P_1 concatenated with P_2 is an x, y -path such that $\varphi_f(P_1^R P_2) > \varphi_g(P_1^R P_2)$. Since the end points of this path belong to directed cycles in D_f , their colours are fixed. Let z be an internal vertex of the path. If z can be recoloured by α it is easy to check the sum of the weight of the two path edges incident with z does not change. (For example, if both edges are directed in the direction of traversal of the path, one increases by α and the other decreases by α . The analyses for the other possible orientations of the two edges are similar.) \square

In [7], fixed vertices are found by successively deleting sources and sinks from D_f . Thus vertices belonging to a directed cycle will be fixed, but also vertices on a directed path between two directed cycles are also fixed. In our work, we identify fixed vertices of the former type using strongly connected components of D_f and vertices of the latter type by the weight of paths between strongly connected components (our third obstruction) as directed paths in D_f have the smallest possible weight over all edge labellings.

In Step 3 of the algorithm, we have recoloured vertices to obtain a colouring f_n such that $\varphi_{f_n} = \varphi_g$. By Corollary 8, $f_n(v) = g(v) + k$ for some constant k . If D_f contains any directed cycles, then there are fixed vertices under f . In Step 1 we have checked that $f(v) = g(v)$ for fixed vertices, so we can conclude $k = 0$ and $f_n = g$. On the other hand, if $k \neq 0$, then there are no directed cycles in D_f . Applying Proposition 9 with $X = V(G)$ shows we can recolour the vertices one at a time to reconfigure f_n to g .

The following theorem is immediate from the algorithm.

Theorem 13. *Let $2 \leq p/q < 4$. Suppose f and g are (p, q) -colourings of a graph G , \vec{G} is an orientation of G and φ_f, φ_g are the edge labellings of G obtained from f and g . Then f recolours to g if and only if:*

1. *the same set of vertices are fixed under f and g and $f(v) = g(v)$ for all fixed vertices;*
2. *for each cycle C of G , $\varphi_f(C) = \varphi_g(C)$; and*
3. *for each path P whose end points are fixed, $\varphi_f(P) = \varphi_g(P)$.*

Moreover, one can find a recolouring sequence from f to g or an obstruction of the above type in polynomial time.

At each iteration of the algorithm we recolour $O(|V(G)|)$ vertices and increase X_e by at least one vertex. Once a vertex is in X_e it never leaves. Thus we do $O(|V(G)|)$ recolouring steps.

The Recolouring Algorithm

Let $2 \leq p/q < 4$.

Input: A graph G and two (p, q) -colourings f, g .

Output: A recolouring sequence from f to g or an obstruction to recolouring.

1. Find the strongly connected components of D_f . For each vertex v belonging to a nontrivial strongly connected component, verify $f(v) = g(v)$. If for some such v , $f(v) \neq g(v)$, then answer NO, return a directed cycle to which v belongs, and STOP.
2. Construct a spanning tree T rooted at a vertex u . Partition $V(G)$ into three sets: X_ℓ, X_e, X_s consisting of those vertices v whose (u, v) -path in T has weight larger, equal, smaller under φ_f versus φ_g respectively. Repeat until $X_\ell \cup X_s = \emptyset$:
 - (a) If $X_\ell \neq \emptyset$, then let $X = X_e \cup X_s$ and $\overline{X} = X_\ell$. (If $X_\ell = \emptyset$ and $X_s \neq \emptyset$ apply an analogous process reversing their roles.) For each $e \in \partial^+(X)$ (resp. $\partial^-(X)$), verify $\varphi_f(e) > \varphi_g(e)$ (resp. $\varphi_f(e) < \varphi_g(e)$). If some edge fails this test, then answer NO, return a cycle with different weights under φ_f and φ_g , and STOP.
 - (b) If $D_f[X]$ and $D_f[\overline{X}]$ both contain nontrivial strongly connected components, then answer NO, return a path P between two fixed vertices with $\varphi_f(P) \neq \varphi_g(P)$, and STOP.
 - (c) Let $\alpha = \min_{e \in \partial(X)} |\varphi_f(e) - \varphi_g(e)|$. Recolour the vertices of X (or \overline{X}) by 1 using a sequence of single vertex recolourings. Repeat this recolouring by X (by 1) α times.
 - (d) Update the path weights in T and the sets X_ℓ, X_e, X_s .
3. At this point $\varphi_f = \varphi_g$. If $f(u) = g(u)$, then answer YES, return the sequence of recolourings, and STOP. Otherwise, we know G contains no directed cycles and we can topologically sort $V(G) : v_1, v_2, \dots, v_n$. Increase (or decrease) the colour of all the vertices by 1 in the order v_n to v_1 . Repeat this until $f(u) = g(u)$. Answer YES, return the recolouring sequence, and STOP.

Figure 1: The Recolouring Algorithm

Corollary 14. *Let $2 \leq p/q < 4$. Suppose f and g are (p, q) -colourings of a graph G . If f recolours to g , then there is a reconfiguration sequence of length $O(|V(G)|^2)$ which certifies this.*

3 The PSPACE-complete Cases: $p/q \geq 4$

3.1 Overview

As in the previous section, all addition and subtraction involving elements of $\{0, \dots, p-1\}$ is viewed modulo p . We say that $i, j \in \{0, \dots, p-1\}$ are *compatible* if they are adjacent in $G_{p,q}$. For notational convenience, let us define $k := \lfloor p/q \rfloor$ and $r := p - kq$.

Consider, for a moment, the case $r = 0$. If $q = 1$, then the complexity is PSPACE-complete by Theorem 1. Otherwise, let $\gamma : \{0, \dots, k-1\} \rightarrow \{0, \dots, p-1\}$ be defined by $\gamma(i) := qi$ and $\phi : \{0, \dots, p-1\} \rightarrow \{0, \dots, k-1\}$ be defined by $\phi(j) := \lfloor j/q \rfloor$. It is not hard to see that γ is a homomorphism from K_k to $G_{p,q}$ and that ϕ is a homomorphism from $G_{p,q}$ to K_k . Also, given an instance (G, f, g) of the k -RECOLOURING problem, we have that f reconfigures to g if and only if γf reconfigures to γg as (p, q) -colourings. Therefore, the (p, q) -RECOLOURING problem is PSPACE-complete if $p/q \geq 4$ and $r = 0$. Thus while the proofs below work when $r = 0$, to avoid trivialities we assume that $r \geq 1$.

Given an instance (G, f, g) of the k -RECOLOURING problem, we will construct an instance (G', α, β) of the (p, q) -RECOLOURING problem in polynomial time such that:

- $|V(G')| = O(|V(G)| + |E(G)|)$ and
- f reconfigures to g if and only if α reconfigures to β .

It will follow from Theorem 1 that the reconfiguration problem for (p, q) -colourings is PSPACE-complete, thereby completing the proof of Theorem 3.

We give a brief outline of the ideas behind the construction before moving into the finer details. The first step is to divide the set $\{0, \dots, p-1\}$ into k intervals which we will, in some sense, treat as k separate colours. Define

$$S_0 := [0, q + r - 1], \text{ and}$$

$$S_i := [iq + r, (i+1)q + r - 1] \text{ for } 1 \leq i \leq k-1.$$

It is clear that S_i and S_j are disjoint for $i \neq j$ and that $\bigcup_{i=0}^{k-1} S_i = \{0, \dots, p-1\}$. Let $\gamma : \{0, \dots, k-1\} \rightarrow \{0, \dots, p-1\}$ be the function which maps i to the left endpoint of S_i for $0 \leq i \leq k-1$. For $i \neq j$, it is easily observed that the left endpoint of S_i is adjacent to the left endpoint of S_j in $G_{p,q}$. Therefore, given any k -colouring $f : V(G) \rightarrow \{0, \dots, k-1\}$, the composition γf is a (p, q) -colouring.

This observation guides part of our construction. The graph G' will contain G as an induced subgraph and the (p, q) -colourings α and β will be defined so that their restrictions to G will be equal to γf and γg , respectively. Now, if it is possible to reconfigure α to β

in such a way that none of the intermediate colourings map a pair of adjacent vertices of G to the same set S_i , then we immediately obtain a reconfiguration sequence taking f to g . That is, to obtain a sequence of k -colourings taking f to g , one could simply compose each (p, q) -colouring of the sequence with the function $\varphi : \{0, \dots, p-1\} \rightarrow \{0, \dots, k-1\}$ which maps every vertex of S_i to i for $0 \leq i \leq k-1$. Notice that, for $i \neq 0$, the set S_i is an independent set in $G_{p,q}$ and therefore no adjacent pair is ever mapped to S_i . Thus, all that we need to worry about is that some of the intermediate colourings may map two adjacent vertices of G to S_0 . To remedy this, we will add some structures (or “gadgets”) to G' which will forbid adjacent vertices of G from mapping to S_0 .

To this end, we start by adding a copy of $G_{p,q}$ disjoint from G with vertex set $\{y_0, \dots, y_{p-1}\}$, where $y_i y_j$ is an edge whenever $q \leq |i - j| \leq p - q$. We extend the (p, q) -colourings α and β to $\{y_0, \dots, y_{p-1}\}$ by setting $\alpha(y_i) = \beta(y_i) = i$ for all i . It is clear that each vertex y_i is fixed in α and β and, moreover, in any (p, q) -colouring which reconfigures to α or β .

Now, for each edge uv of G , we will add two *forbidding paths* P_{uv} and P_{vu} from u to v to G' which are internally disjoint from $V(G) \cup \{y_0, \dots, y_{p-1}\}$ and one another, as well as from every other such path. These paths restrict the colour pairs which can appear on u and v during a reconfiguration process. We achieve this by assigning to each interval vertex of the path, a list of allowed colours for that vertex, i.e. we use a list colouring. This is accomplished by joining the internal vertices of the paths to specific subsets of $\{y_0, \dots, y_{p-1}\}$. For example consider the case $p = 18, q = 4$ which we explore below. Suppose we want a path vertex x to always be coloured with an element of the list $\{4, 5, \dots, 11\}$. Then it suffices to join x to the vertices $\{y_{15}, y_{16}, y_{17}, y_0\}$. As we will show, the colours in the lists forbids u and v from mapping to S_0 . Also, once we describe our construction in detail, it will be clear that the length of the forbidding paths depends only on p and q , and so $|V(G')| = O(|V(G)| + |E(G)|)$.

The difficulty now comes in proving that if f reconfigures to g , then α reconfigures to β . Specifically, given a reconfiguration sequence $(h_i)_{i=1}^s$ taking f to g , we need that the lists assigned to the internal vertices of the forbidding paths are flexible enough that we can use $(h_i)_{i=1}^s$ to obtain a reconfiguration sequence taking α to β . This will be proved using a moderate amount of case analysis at the end of the section. Before moving on, we remark that the general strategy of using some sort of forbidding paths in which the internal vertices are confined to lists was also used by Bonsma and Cereceda [3] (in a somewhat different manner) to prove that the reconfiguration problem for k -colourings is PSPACE-complete for $k \geq 4$.

3.2 Defining the Forbidding Paths

Let uv be an edge of G . We will now describe our construction of the forbidding path $P_{uv} = ux_0^{uv} x_1^{uv} \dots x_t^{uv} v$ from u to v , including the definition of the lists assigned to the internal vertices $x_0^{uv}, \dots, x_t^{uv}$. As mentioned before, the construction of G' also includes a path P_{vu} from v to u which is defined similarly.

First, the construction of the lists depends on the sequence $q, 2q, 3q, \dots, r$. Define t to be the smallest positive integer such that $(t+1)q \equiv r \pmod{p}$. Now, the *lists* for the internal

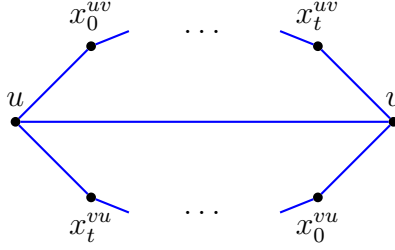


Figure 2: Forbidding paths attached to each edge uv

vertices of P_{uv} are intervals of $G_{p,q}$ and are defined as follows:

$$L(x_0^{uv}) = L(x_t^{uv}) := [p-1, 2q-1],$$

$$L(x_i^{uv}) := [iq, (i+2)q-1] \text{ for } 1 \leq i \leq t-1.$$

As stated in the outline, in order to enforce these lists we add a copy of $G_{p,q}$ on vertex set $\{y_0, \dots, y_{p-1}\}$ and join the vertices of $P_{uv} \setminus \{u, v\}$ to the appropriate vertices of this set. Specifically, we join the vertices x_0^{uv} and x_t^{uv} to $\{y_j : j \in [3q-1, p-q-1]\}$ and join x_i^{uv} to $\{y_j : j \in [(i+3)q-1, (i-1)q]\}$ for $1 \leq i \leq t-1$.

Figure 3 contains a useful tabular representation of the lists, and a specific case $(p, q) = (18, 4)$ is depicted in Figure 4. In both of these figures, the colours are laid out so that below colour c in the table is the colour $c+q$. This gives the following proposition.

Proposition 15. *When colouring the path $x_0^{uv}, x_1^{uv}, \dots, x_t^{uv}$ the colours used (one per row) must be directly below or to the right of the preceding vertex, and any such sequence of colours from the table gives a good colouring of the path with the exception of $L(x_{t-1}^{uv})$ and $L(x_t^{uv})$. In this case colours in the interval $[r-2q, r-1]$ of $L(x_{t-1}^{uv})$ are compatible with colours below and to the right in the interval $[p-1, q+r]$ of $L(x_t^{uv})$. Similarly, the interval $[-q-1, r-1]$ is compatible with colours below and to the right of $[p-1, 2q-1]$.*

Proof. For $1 \leq i \leq t-2$ vertex x_i^{uv} receives a colour from the interval $[iq, (i+2)q-1]$ and x_{i+1}^{uv} receives a colour from $[(i+1)q, (i+3)q-1]$. Observe iq is compatible with all colours in the interval $[(i+1)q, (i-1)q]$. Since $p+(i-1)q \geq (i+3)q-1$, we have iq is compatible with every colour on the list for x_{i+1}^{uv} . The colour $iq+1$ is compatible with $[(i+1)q+1, (i-1)q+1]$ which include all colours below and to the right of $iq+1$. Continuing one sees that $(i+2)q-1$ is compatible with $[(i+3)q-1, (i+1)q-1]$ which includes only the last vertex of the list for x_{i+1}^{uv} . Thus the proposition holds for $1 \leq i \leq t-2$. It is straightforward to verify the case $i=0$ as well. Consider now $i=t-1$. The argument is similar; however, we need to consider the two cases in the statement of the proposition. First for a colour $[r-2q, r-1]$, a colour in $[p-1, q+r]$ is compatible if and only if it is below and to the right. (Note the right hand end point $q+r$ is compatible with $r-2q$ if and only if $p+r-2q-q \geq q+r$ or $p \geq 4q$.) A similar statement holds for $[-q-1, r-1]$ and $[p-1, 2q-1]$. The result follows. \square

Finally the vertical bar in the table provides the following information: if vertex u receives colour 0, then only colours to the right of the vertical bar can be used to colour the path P_{uv} .

Vertex	Lists									
x_0^{uv}	$p-1$	0	...	$q-r-1$...	$q-1$	q	...	$2q-1$	
x_1^{uv}		q	...	$2q-r-1$...	$2q-1$	$2q$...	$3q-1$	
x_2^{uv}		$2q$...	$3q-r-1$...	$3q-1$	$3q$...	$4q-1$	
\vdots		\vdots		\vdots		\vdots		\vdots		
x_i^{uv}		iq	...	$(i+1)q-r-1$...	$(i+1)q-1$	$(i+1)q$...	$(i+2)q-1$	
\vdots		\vdots		\vdots		\vdots		\vdots		
x_{t-1}^{uv}		$r-2q$...	$-q-1$...	$r-q-1$	$r-q$...	$r-1$	
x_t^{uv}				$p-1$...	$r-1$	r	...	$q+r-1$... $2q-1$

Figure 3: Assignment of lists to the internal vertices of the forbidding path.

Vertex	Lists									
x_0^{uv}	17	0	1	2	3	4	5	6	7	
x_1^{uv}		4	5	6	7	8	9	10	11	
x_2^{uv}		8	9	10	11	12	13	14	15	
x_3^{uv}		12	13	14	15	16	17	0	1	
x_4^{uv}			17	0	1	2	3	4	5	6 7

Figure 4: Assignment of lists to the internal vertices of the forbidding path for the $(p, q) = (18, 4)$ case.

Before moving on, let us check that the paths P_{uv} and P_{vu} actually do the job that they are meant to do; namely, that they forbid u and v from both being mapped to S_0 .

Proposition 16. *Let uv be an edge of G . If ψ is a (p, q) -colouring of $P_{uv} \cup P_{vu}$ such that the internal vertices of each path are coloured from their lists, then $\psi(u)$ and $\psi(v)$ cannot both be contained in S_0 .*

Proof. Suppose to the contrary that $\psi(u), \psi(v) \in S_0$. In $G_{p,q}$, the only edges contained in S_0 are between vertices in $[0, r-1]$ and vertices in $[q, q+r-1]$. So, without loss of generality, we can assume that

$$\psi(u) \in [0, r-1] \text{ and } \psi(v) \in [q, q+r-1]. \quad (3)$$

However, the fact that $\psi(u) \in [0, r-1]$ implies that $\psi(x_0^{uv}) \in [q, 2q-1]$ (Note that $r-1 < q-1$ and thus $(r-1, p-1)$ is not an edge of $G_{p,q}$.) By our observations above, we see that only colours to the right of the vertical line in Figure 3 can be used to colour P_{uv} . In particular $\psi(x_t^{uv}) \in [r, 2q-1]$. This implies that $\psi(v) \notin [q, q+r-1]$, which contradicts (3) and completes the proof. \square

3.3 Defining α and β on the Forbidding Paths

As we have already mentioned, α and β are defined in such a way that their restrictions to G are equal to γf and γg , respectively, and $\alpha(y_i) = \beta(y_i) = i$ for all i . Next, we will describe the way in which we extend α and β to the vertices of the forbidding paths.

Recall that the vertices of G each receive a colour from $\{0, q + r, 2q + r, \dots, (k - 1)q + r\}$ under α . Given an edge uv of G , the internal vertices of P_{uv} are coloured as follows. If $\alpha(u) \neq 0$ and $\alpha(v) \neq 0$, then colour x_i^{uv} with colour iq for $i = 0, 1, 2, \dots, t - 1$ and colour x_t^{uv} with 0. (These colours correspond to the left hand column in Figure 3 for $1 \leq i \leq t - 1$.) The path P^{vu} is similarly coloured so that $\alpha(x_i^{vu}) = iq$ and $\alpha(x_t^{vu}) = 0$. In particular, $\alpha(x_0^{uv}) = \alpha(x_t^{vu}) = 0$ for all $v \in N(u)$. Note that $(t - 1)q \equiv r - 2q \pmod{p}$ which is compatible with 0. On the other hand, if $\alpha(u) = 0$, then set $\alpha(x_i^{uv}) = (i + 1)q$ for $i = 0, 1, \dots, t$. For P_{vu} , set $\alpha(x_i^{vu}) = iq$ for $i = 0, 1, \dots, t - 1$ and $\alpha(x_t^{vu}) = q$. Observe in this case, the internal vertices around u have colour q , i.e. $\alpha(x_0^{uv}) = \alpha(x_t^{vu}) = q$ for all $v \in N(u)$, and the internal vertices around v have colour r or 0, i.e. $\alpha(x_t^{uv}) = r$ and $\alpha(x_0^{vu}) = 0$ for all $v \in N(u)$.

We similarly extend β using g . A technical detail we will exploit below is that in extending α and β the colour for x_t^{uv} belongs to $\{0, r, q\}$. These colours are compatible with all colours above and to the left in the row for x_{t-1}^{uv} by Proposition 15. We call these colourings of P_{uv} the *standard path colourings*. At the start of each reconfiguration step we assume the paths have a standard path colouring, and the end of each reconfiguration step we ensure the paths have a standard path colouring. The standard path colourings correspond to columns in Figure 3 with possible changes at x_t^{uv} .

By Proposition 16, given any sequence of (p, q) -colourings which reconfigures α to β , we can compose each of these colourings with γ to obtain a sequence of k -colourings taking f to g . This proves the following proposition.

Proposition 17. *Suppose (G', α, β) is an instance of (p, q) -RECOLOURING obtained from (G, f, g) , an instance of k -RECOLOURING, as described above. If α reconfigures to β , then f reconfigures to g .*

3.4 Recolouring G'

To complete the reduction we need to prove that if f reconfigures to g , then α reconfigures to β . Let $(h_i)_{i=1}^s$ be any reconfiguration sequence taking f to g . We show that there is a sequence $(\eta_i)_{i=1}^s$ of (p, q) -colourings of G' such that

- $\eta_1 = \alpha$ and $\eta_s = \beta$,
- the restriction of η_i to G is γh_i for $1 \leq i \leq s$, and
- η_i reconfigures to η_{i+1} for $1 \leq i \leq s - 1$.

Clearly this will prove that α reconfigures to β and complete the proof of Theorem 3.

Before tackling the general case we illustrate our method with the example in Figure 5, in which $(p, q) = (18, 4)$. In this case, the colourings η_i map $V(G)$ to the colours $\{0, 6, 10, 14\}$.

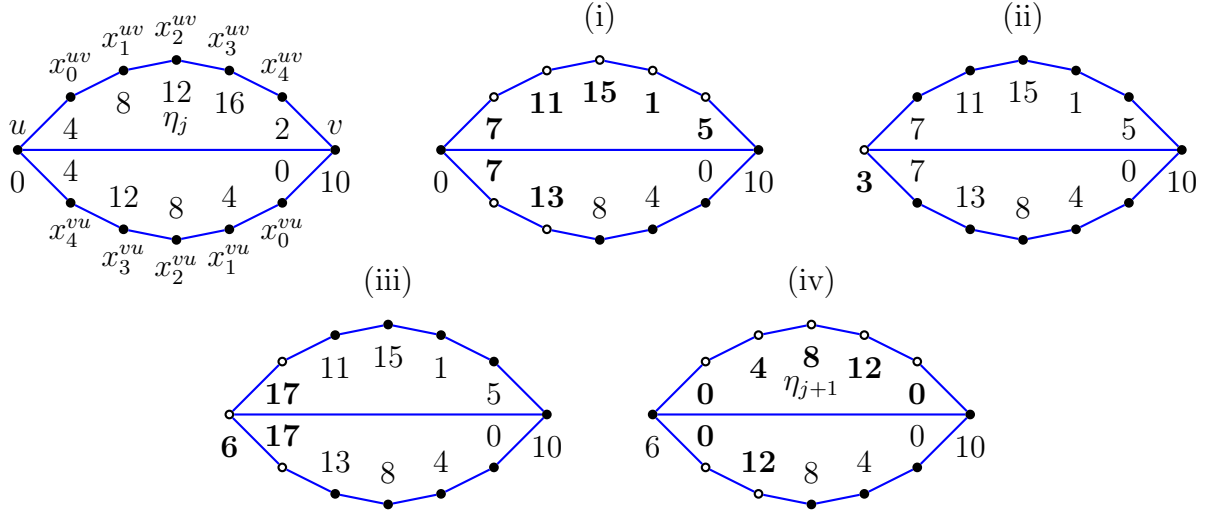


Figure 5: Recolouring u from 0 to 6 in the $(18, 4)$ case.

Let $1 \leq j \leq s - 1$ be fixed and suppose that $(\eta_i)_{i=1}^j$ have been constructed to satisfy the conditions above; our goal is to construct the colouring η_{j+1} and a reconfiguration sequence taking η_j to η_{j+1} .

By definition, h_j and h_{j+1} differ on at most one vertex, say $u \in V(G)$. Suppose, for example that $h_j(u) = 0$ and $h_{j+1}(u) = 1$; this requires us change the colour of u from 0 (its current colour under η_j) to 6 (its desired colour under η_{j+1}) without changing the colour of any other vertex of G . We remark that this is the most involved case, and corresponds to Case (c) in Lemma 18 below. The other cases use similar ideas but are more straightforward. For each vertex $v \in V(G)$ to which u is adjacent, v must receive a colour from $\{2, 3\}$ under h_j to facilitate the recolouring of u , which implies that $\eta_j(v) \in \{10, 14\}$. Each step below is applied for each neighbour v of u , one by one, before moving on to the subsequent step. The recoloured vertices at each step are shown as white vertices in Figure 5 with the new colours indicated in boldface.

- (i) First, recolour x_3^{vu} to 13 and x_4^{vu} to 7. Then, recolour P_{uv} by recolouring x_4^{uv} to 5, x_3^{uv} to 1, x_2^{uv} to 15, x_1^{uv} to 11 and x_0^{uv} to 7. At this point all vertices of $\{x_0^{uv}, x_t^{vu} : v \in N_G(u)\}$ have colour 7.
- (ii) Now, change the colour of u to 3 (temporarily).
- (iii) Next, change the colour of x_0^{uv} to 17 and x_t^{vu} to 17. At this point all vertices of $\{x_0^{uv}, x_t^{vu} : v \in N_G(u)\}$ have colour 17. Recolour u to 6.
- (iv) Recolour the paths to the standard colourings: x_i^{uv} is coloured iq for $i = 0, 1, \dots, t - 1$ and x_t^{uv} is coloured 0; x_{t-1}^{vu} is coloured 12 and x_t^{vu} is coloured 0.

We now describe the recolouring technique in general.

Lemma 18. *Let h and h' be k -colourings of G which differ on a unique vertex u and let η be a (p, q) -colouring of G' such that*

- η is the standard path colouring for all paths P_{uv} ,
- $\eta(y_i) = i$ for all i , and
- the restriction of η to G is γh .

Then there exists a (p, q) -colouring η' of G' such that η' is the standard path colouring for all paths P_{uv} , the restriction of η' to G is $\gamma h'$, and η reconfigures to η' .

Proof. We provide a reconfiguration sequence which takes η to a colouring η' with the desired properties. The proof is divided into cases. In each case, each of the reconfiguration steps is done for every neighbour v of u , one by one, before moving on to the subsequent step.

As mentioned above, when extending α from $V(G)$ to $V(G')$, the colour for x_t^{uv} is selected from the interval $[p-1, q+r]$. In some cases below we may assign x_t^{uv} a colour outside of this interval (but still on its list). However, the assignment will be compatible with the current colour of x_{t-1}^{uv} and at the end of the case the colour of x_t^{uv} will be in $[p-1, q+r]$.

Let uv be an edge in G and assume the paths P_{uv} and P_{vu} have standard path colourings.

Case (a): $\eta(u), \eta'(u) \in \{q+r, 2q+r, \dots, (k-1)q+r\}$ In standard path colourings $\eta(x_0^{uv}), \eta(x_t^{vu}) \in \{0, r\}$ for each $v \in N(u)$. The colour $\eta'(u)$ is compatible with both 0 and r , so we may change the colour of u to $\eta'(u)$. The paths still have standard path colourings.

Case (b): $\eta(u) = 0, \eta'(u) \in \{2q+r, 3q+r, \dots, (k-1)q+r\}$

- (i) By the definition of standard path colourings, $\eta(x_0^{uv}) = \eta(x_t^{vu}) = q$. Change the colour of u to $\eta'(u)$.
- (ii) Recolour x_t^{vu} to 0. The path P_{vu} now has a standard path colouring.
- (iii) Recolour x_i^{uv} to iq for $i = 0, 1, \dots, t-1$ and x_t^{uv} to 0. The path P_{uv} now has a standard path colouring.

Case (c): $\eta(u) = 0, \eta'(u) = q+r$

- (i) We begin observing that $h(u) = 0$ and $h'(u) = 1$, which implies that $h(v) \in \{2, \dots, k-1\}$. Thus, $\eta(v) \in \{2q+r, 3q+r, \dots, (k-1)q+r\}$. For $i = t, \dots, 0$, in order, recolour x_i^{uv} to $(i+2)q-1$.
- (ii) Change the colours of x_{t-1}^{vu} to $-q-1$, and x_t^{vu} to $2q-1$. These colours are compatible with each other, and by Proposition 15, the colour of x_{t-1}^{vu} and x_{t-2}^{vu} are compatible as well.
- (iii) At this point all vertices of $\{x_0^{uv}, x_t^{vu} : v \in N_G(u)\}$ have colour $2q-1$. Recolour u to $q-1$ (temporarily). Note that this colour is compatible with all neighbours v of u in G since $\eta(v) \in \{2q+r, 3q+r, \dots, (k-1)q+r\}$.

- (iv) Recolour x_0^{uv} to $p - 1$ and x_t^{vu} to $p - 1$.
- (v) At this point all vertices of $\{x_0^{uv}, x_t^{vu} : v \in N_G(u)\}$ are coloured $p - 1$. Change the colour of u to $q + r$.
- (vi) Recolour x_{t-1}^{vu} to $r - 2q$ and x_t^{vu} to 0 . Recolour x_i^{uv} to iq for $i = 0, 1, \dots, t - 1$ and x_t^{uv} to 0 . The paths now have the standard path colourings.

Case (d): $\eta(u) \in \{2q + r, 3q + r, \dots, (k - 1)q + r\}, \eta'(u) = 0$

- (i) Change the colour of x_t^{vu} to q , and recolour x_i^{uv} to $(i + 1)q$ for $i = t, t - 1, \dots, 0$.
- (ii) At this point all vertices of $\{x_0^{uv}, x_t^{vu} : v \in N_G(u)\}$ have colour q . Now change the colour of u to 0 .
- (iii) The paths have standard path colourings.

Case (e): $\eta(u) = q + r, \eta'(u) = 0$

- (i) Recolour x_t^{vu} to $p - 1$ and x_0^{uv} to $p - 1$.
- (ii) At this point all vertices of $\{x_0^{uv}, x_t^{vu} : v \in N_G(u)\}$ have colour $p - 1$. Change the colour of u to $q - 1$ (temporarily). Note that this colour is compatible with all neighbours v of u in G since $\eta(v) \in \{2q + r, 3q + r, \dots, (k - 1)q + r\}$.
- (iii) Since $\eta(v) \in \{2q + r, 3q + r, \dots, (k - 1)q + r\}$, for $i = t, \dots, 0$ we can recolour x_i^{uv} to $(i + 2)q - 1$.
- (iv) Change the colour of x_{t-1}^{vu} to $-q - 1$, x_t^{vu} to $2q - 1$ and u to 0 .
- (v) Finally, recolour x_i^{uv} to $(i + 1)q$ for $i = 0, 1, \dots, t$. Recolour x_t^{vu} to q , and x_{t-1}^{vu} to $r - 2q$. The paths have the standard path colourings.

This completes the proof of the lemma, and of Theorem 3. □

4 Cycles in Graphs of Large Chromatic Number

A result of Chen and Saito [8] says that graphs without cycles of length divisible by three are 2-degenerate. In particular, it follows that such graphs are 3-colourable. Recently, Wrochna discovered a short and elegant proof of the 3-colourability result using ideas from [7]. He has decided not to publish it himself, but has given us permission to include it here.

Theorem 19 (Chen and Saito [8]). *If G contains no cycle of length $0 \bmod 3$, then G is 3-colourable.*

Proof (Wrochna). Suppose that the statement is false and let G be a counterexample for which $|E(G)|$ is minimum. Define $G' := G - e$ where $e = uv$ is an edge of G . By hypothesis, G' admits a 3-colouring, say $f : V(G') \rightarrow \{0, 1, 2\}$. If $f(u) \neq f(v)$, then we are done. So we assume, without loss of generality, that $f(u) = f(v) = 0$.

Now, let $g : V(G) \rightarrow \{0, 1, 2\}$ be defined by $g(x) = f(x) + 1 \bmod 3$ for all $x \in V(G)$. It is clear that $\varphi_f(C) = \varphi_g(C)$ for all cycles C in G' . Also, since G' has no cycles of length $0 \bmod 3$, there are no directed cycles in D_f or D_g , and therefore no fixed vertices either. Thus, by Theorem 13, f can be reconfigured to g . Let h be the first colouring of the reconfiguration sequence such that $h(u) \neq 0$ or $h(v) \neq 0$. Since the reconfiguration sequence colours only one vertex in each step, it is clear that h is a 3-colouring of G , which completes the proof. \square

A closer look at the proof shows that it actually yields something slightly stronger: *If G contains an edge e such that $G - e$ has no cycle of length $0 \bmod 3$, then $\chi(G) \leq 3$.* Using a similar strategy, we can generalize this to k -colourings.

Theorem 20. *If G contains an edge e such that $G - e$ contains fewer than $\frac{(k-1)!}{2}$ cycles of length $0 \bmod k$, then $\chi(G) \leq k$.*

Proof. We proceed by induction on $|E(G)|$, where the case $|E(G)| = 0$ is trivial. Let $e = uv$ be an edge of G such that $G' := G - e$ contains fewer than $\frac{(k-1)!}{2}$ cycles of length $0 \bmod k$. Then, by the inductive hypothesis, G' admits a k -colouring $f : V(G') \rightarrow \{0, \dots, k-1\}$. If $f(u) \neq f(v)$, then we are done. So we assume, without loss of generality, that $f(u) = f(v) = 0$.

Let π be a permutation of $\{0, \dots, k-1\}$ with $\pi(0) = 0$. Let $F_{f,\pi}$ be an oriented spanning subgraph of G' where $F_{f,\pi}$ contains an arc \overrightarrow{xy} if $f(x) = \pi(i)$ and $f(y) = \pi(i+1)$ for some i (addition is modulo k). Let S_f be the set of vertices in G which can be reached by an oriented path in $F_{f,\pi}$ starting at u . Suppose that $F_{f,\pi}[S_f]$ does not contain a directed cycle. Then $F_{f,\pi}[S_f]$ contains a sink x . Recolour x from say $\pi(j)$ to $\pi(j+1)$, i.e. recolour x to $\pi(\pi^{-1}(f(x)) + 1)$. This results in a proper k -colouring f' of G' in which $F_{f',\pi}[S_{f'}] = F_{f,\pi}[S_f] - x$. Repeating this procedure, one eventually reaches a situation where either u or v is a sink and so the colour of either u or v is changed; hence, there is a first k -colouring g where $g(u) \neq g(v)$ and we are done.

Therefore, $F_{f,\pi}$ must contain a directed cycle for every permutation π with $\pi(0) = 0$. Such a cycle must have vertex colours $\pi(i), \pi(i+1), \pi(i+2), \dots, \pi(i-1), \pi(i)$, and thus have length $0 \bmod k$. On the other hand, let C be a cycle of length $0 \bmod k$ in G' . The colours of C under f have the following property: there is a vertex of colour 0, starting at that vertex and traversing the cycle we see all k colours in the first k vertices, and each successive block of k vertices is coloured with the same colours in the same order. In other words, the first k vertices define a permutation π such that $\pi(0) = 0$, and C is a directed cycle in $F_{f,\pi}$. Observe if we traverse C in the opposite direction we obtain a second permutation π' where C is directed in $F_{f,\pi'}$. Moreover if C is a directed cycle in $F_{f,\sigma}$ with $\sigma(0) = 0$, then σ must be π or π' . Since G' has fewer than $(k-1)!/2$ cycles of length $0 \bmod k$, there must be some permutation π such that $F_{f,\pi}$ is acyclic. The result follows. \square

Corollary 21. *If $\chi(G) > k$, then G contains at least $\frac{(k-1)!}{2}$ cycles of length $0 \bmod k$.*

The complete graph of order $k+1$ has precisely $\frac{(k+1)(k-1)!}{2}$ cycles of length $0 \bmod k$, and so Corollary 21 is within a factor $k+1$ of being tight. We wonder whether K_{k+1} contains the fewest cycles of length $0 \bmod k$ among all non- k -colourable graphs.

Conjecture 22. *If $\chi(G) > k$, then G contains at least $\frac{(k+1)(k-1)!}{2}$ cycles of length $0 \bmod k$.*

The case $k = 3$ may be particularly instructive: *Is it true that every graph G with $\chi(G) \geq 4$ contains at least 4 cycles of length $0 \bmod 3$?* If Conjecture 22 turns out to be false, then it would still be interesting to determine the minimum number of cycles of length $0 \bmod k$ in a graph of chromatic number greater than k .

More generally, one could investigate the minimum number cycles of length $r \bmod k$ in a graph of chromatic number at least, say, $f(r, k)$. With regards to the *existence* of such cycles, Chen, Ma and Zang [9] proved that any graph with chromatic number greater than k must contain a cycle of length $r \bmod k$ for $r \in \{0, \dots, k-1\} \setminus \{2\}$ and that any graph of chromatic number greater than $k+1$ must also contain a cycle of length $2 \bmod k$. Dean, Lesniak and Saito [12] proved that if $\chi(G) \geq 4$, then G has a cycle of length $0 \bmod 4$. Similar problems for *induced* cycles are very well studied but usually, in this setting, the function f also depends on the size of the largest clique in G ; see, e.g. [2, 11, 10, 13, 19, 20, 21].

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